Performance Comparison of Reconstruction from Non-uniform Samples Using Sinc Interpolation and Method of Spline

Apurva H. Ghate[1], K. B. Khanchandani[2]

1,2 Department of Electronics & Telecommunications
Shri Sant Gajanan Maharaj College of Engineering Shegaon - 444203 Dist. Buldana (Maharashtra) India
e-mail : apurva.ghate@gmail.com

Abstract— We discuss the reconstruction of functions from their samples at non-uniformly distributed locations using two methods of approximation i.e. sinc interpolation and method of spline. Also We simulate reconstruction results for both the methods and compared performance of them based on the errors like mean squared error and mean absolute deviation.

Keywords : Non-uniform sampling, Signal reconstruction, Sinc interpolation, B-splines.

1. INTRODUCTION

According to Shannon sampling theorem [14] a continuous-time signal can be exactly reconstructed from its uniform Discrete time (DT) samples if the signal is band limited and if it follows the Nyquist criteria. Similar results are available for non-uniform sampling but the operations involved are far more complicated than the uniform sampling. Non-uniform sampling can be seen as extension of uniform sampling, with constraining samplings on non-uniform grid. In general uniform reconstruction is done by using time invariant system but in real life samples are non-uniform and hence existing technology is of no use. As it is time variant system and samples are arbitrarily placed, it is difficult to perfectly reconstruct the samples.

In many practical applications, sampling occurs at non-uniform time instances [2]. If the average sampling rate is at least equal to the Nyquist rate, then the band limited continuous-time signal is uniquely determined by these non-uniform samples [6]. Although direct reconstruction using continuous-time interpolation functions is, in principle, possible [11], [16] a practical implementation of these functions with high precision is computationally difficult. Non-uniform sampling arises in the situations when we transmit signal the loss of information takes place at the receiver. Jittered errors are introduced at receiver because of the environmental disturbances. Also in ADCS where signals are multiplexed and expected to send simultaneously, because of asynchronous interleaving clock pulses samples at each channel lead to recurrent non-uniform sampling. Another potential application of non-uniform sampling involves sampling based on time varying signal parameters. Signals can be characterized by a set of parameters. These signals are usually more efficiently represented if information about these parameters is factored into the sampling instants. One such example is, signals with fluctuations in their local bandwidths. An efficient representation of such signals can be obtained by sampling them based on their local bandwidths or characteristics, thus leading to a lower overall sampling rate [19]. Another motive we give importance to non uniform sampling as all biomedical and biological signals are non-uniform in nature. Biological receptors in eye, ear are samples data non-uniformly. ECG, EEG, EMG are also non-uniform in nature. It also has its application in astrological signal processing, image processing, RADAR etc.

ISSN 2229-5518

From all the above discussion we conclude that practical non-uniform sampling can be intentional or non-intentional. It donot degrade systems always but sometimes it is deliberate and advantageous also. Until now the perfect reconstruction is not possible. So, various approximation methods are introduced by different researchers. Exact reconstruction of a bandlimited continuous-time signal from nonuniform samples is based on Lagrange interpolation. For the case of uniform sampling, Lagrange interpolation reduces to sinc interpolation and can be approximated with well designed lowpass filtering. When the sampling grid is not uniform, Lagrange interpolation is more difficult. In this paper we consider sinc interpolation of nonuniform samples as a way to approximately reconstruct the continuous-time signal.

Another approximation we can use is method of B-splines. In this method we are using traditional approach of using the interpolation kernel to satisfy interpolation constraints, which requires the kernel to vanish at all sampling points except the origin.B-splines are also evaluated quite easily, using their definition as a divided difference of the truncated power function. Unfortunately, such calculations are ill-conditioned, particularly for partitions of widely varying interval lengths, as is indicated by the fact that special measures have to be taken in case of coincident knots.The most popular generalized inetrpolation function due to their excellent interpolation properties and short support, are uniform B-splines.Recontruction using uniform B-splines and its results are given in this paper.

In this paper we will consider reconstruction using sinc interpolation and method of splines, compare both depending on the simulation results and errors we will obtain so that we can compare the best among the two.

2. RECONSTRUCTION USING SINC INTERPOLATION

As we have already discussed, reconstruction from non uniform samples is the way complicated than that of the uniform sampling as it needs the time variant system to deal with. Hence to reconstruct signals from non uniform samples, a time dependent signal representation has been developed. Framework is used to review some of the sampling theorems presented in [9] and [12]. In a classic paper on non-uniform sampling of band-limited signals [20], Yen introduced several reconstruction theorems. This deals with finite number of non-uniform samples on an otherwise uniform grid, a single gap in uniform sampling and recurrent non-uniform sampling. There are different approaches that can be used for reconstruction as proposed in ([6], [10], [13], [15], [18]) which follows the recurrent non-uniform sampling approach.

In this paper we will use approach contributed by Yao and Thomas [21]. They used Lagrange interpolation functions for the reconstruction of band limited signals from non-uniform samples. It is shown there that a finite-energy signal $x=(t)$ band limited $\pm\pi/T_N$ can be reconstructed from its non-uniform samples $x(t_n)$ using Lagrange interpolation when the sampling instants t_n do not deviate by more than $T_N/4$ from a uniform grid with spacing of T_N .

Specifically,

$$\text{if } |t_n - nT_N| \le d < T_N/4, \quad \forall n \in Z \tag{1}$$

Then

$$x(t) = \sum_{n=-\infty}^{\infty} x(t_n) l_n(t), \tag{2a}$$

where

$$l_{n(}(t) = \frac{G(t)}{G'(t_n)(t-t_{n)})}, \tag{2b}$$

$$G(t) = (t-t_0) \prod_{\substack{k=-\infty,\\ k\neq 0}}^{\infty} \left(1 - \frac{t}{t_k}\right), \tag{2c}$$

and

$$G'(t) \hat{=} \frac{dG(t)}{dt}.$$

Interpolation using equations (2a, 2b, 2c) is referred to as Lagrange interpolation. This is based on theorem by Levinson, where he states, if condition in equation (1) is satisfied and

$$\{l_{n(}(t)\} \xrightarrow{F.T.} \{L_n(\Omega)\}$$

these functions are band limited and formed a sequence bi-orthogonal to $\{e^{j\Omega t_n}\}$ over $\left[\frac{-\pi}{T_N}, \frac{\pi}{T_N}\right]$.

$$\text{i.e.,} \quad L_n(\Omega) = \int_{-\infty}^{\infty} l_n(t) e^{-j\Omega t} dt = 0 \quad , \quad |\Omega| > \frac{\pi}{T_N} \tag{3}$$

and

$$\frac{1}{2\pi} \int_{-\frac{\pi}{T_N}}^{\frac{\pi}{T_N}} L_n(\Omega) e^{j\Omega t_k} d\Omega = l_n(t_k) = \delta[n-k] \tag{4}$$

Equation (4) utilizes the interpolation condition of the Lagrange kernel which ensures that the property of consistent resampling is upheld, i.e., that sampling the reconstructed signal on the non-uniform grid $\{t_n\}$ yields the original samples $\{x(t_n)\}$. Note that expressing $L_n(\Omega)$ in (4) as the Fourier transform of $l_n(t)$ results in bi-orthogonality of the sequences $\{l_n(t)\}$ and $\{\sin c(\pi/T_N(t-t_n))\}$.

i.e., $$\int_{-\infty}^{\infty} l_n(t)\sin c(\pi/T_N(t-t_k)/T_N\,dt = \delta|n-k| \tag{5}$$

The non-uniform sampling theorem in [21] states that a signal $x(t)$ belonging to the class of functions band-limited to $w_0 = \pi/T$ rad/s, can be represented as a series expansion.

We denote by $x[n]$ a sequence of non-uniform samples of $x(t)$,

i.e., $x[n] = x(t_n)$,

where $\{t_n\}$ represents a non-uniform grid which can be given as

$$t_n = nT + \in_n \tag{6}$$

where $\in_n$ are the deviations from uniform grid.

We can define

$$G_n(t) = \frac{\sin\left(\frac{\pi}{T}(t-\in_n)\right)}{\pi/T}. \tag{7}$$

The derivative of above equation is

$$G'_n(t) = \cos\left(\frac{\pi}{T}(t-\in_n)\right)$$

which can be evaluated at t_n, has the value $G'(t_n) = (-1)^n$. Let the scaled Hilbert transformer remain as $h(t) = 1/t$. This gives approximation as

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(t_n)\frac{G_n(t)}{(t-t_n)G'_n(t_n)} \tag{8}$$

$$= \sum_{n=-\infty}^{\infty} x(t_n)\frac{\sin\left(\frac{\pi}{T}(t-t_n)\right)}{\frac{\pi}{T}(t-t_n)}$$

$$= \sum_{n=-\infty}^{\infty} x(t_n).\sin c\left(\frac{\pi}{T}(t-t_n)\right). \tag{9}$$

The sinc kernel $l_n(t) = \dfrac{\sin\left(\frac{\pi}{T}(t-t_n)\right)}{\frac{\pi}{T}(t-t_n)}$ kernel does not satisfy the interpolation property which states that

$$l_n(t_k) = \begin{cases} 1 & , \quad n = k \\ 0 & , \quad n \neq k \end{cases}$$

Hence for non-uniform sample instants, this approximation method causes inter-symbol interference and does not result in consistent sampling.

For the special case the deviations from uniform grid are equal and constant such that $\in_n = \in$, Equation (9) becomes

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(nT+\in)\frac{\sin\left(\frac{\pi}{T}(t-nT-\in)\right)}{\frac{\pi}{T}(t-nT-\in)} \tag{10}$$

and this perfectly reconstructs $x(t)$. Another way of forming an approximation is to assume that the samples lie on the uniform grid and through sinc interpolation,

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(t_n)\frac{\sin\left(\frac{\pi}{T}(t-nT)\right)}{\frac{\pi}{T}(t-nT)}. \tag{11}$$

Fig. 1 shows the reconstruction from non uniform samples by using equation (10) which is obtained by putting equation (6) in equation (9).

Fig. 2 shows reconstruction using sinc interpolation if $\in_n$ deviations are constant. This kind of sampling is considered equal to the uniform sampling.

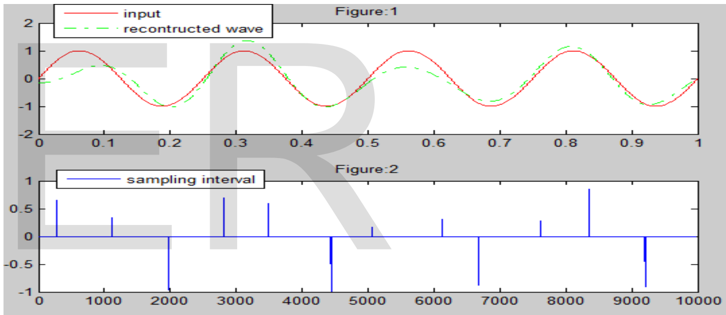

Fig. 1 Reconstruction using sinc interpolation

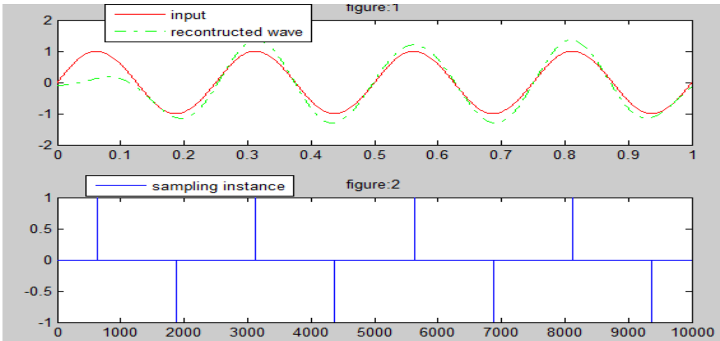

Fig. 2 Reconstruction from uniform samples using sinc interpolation

3. RECONSTRUCTION USING METHOD OF SPLINES :

The non-uniform reconstruction problem with restrictions on the samples locations, usually a density constraint or a maximum gap $x[n+1]-x[n]$, so as to achieve perfect reconstruction of an unknown signal from its non-uniform samples ([1], [7], [8]). However, in many practical applications, this is far too restrictive. If these conditions are not met, perfect reconstruction cannot be guaranteed; hence the problem becomes ill-posed. One way to handle this issue is to adopt a variety of approach as we used method of spline.

In computational dealings with splines, the question of representation is of primary importance. For splines of fixed order on a fixed partition, this is a question of choice of basis, since such splines form a linear space. Only three kinds of bases for spline spaces have actually been given serious attention; those consisting of truncated power functions, of cardinal splines, and of B-splines. Truncated power bases are known to be open to severe il conditioning, while cardinal splines are difficult to calculate. By contrast, bases consisting of B-splines are well-conditioned, at least for orders < 20.Such bases are also local in the sense that at every point only a fixed number (equal to the order) of B-splines is nonzero. B-splines are also evaluated quite easily, using their definition as a divided difference of the truncated power function. Unfortunately, such calculations are ill-conditioned, particularly for partitions of widely varying interval lengths, as is indicated by the fact that special measures have to be taken in case of coincident knots. The condition of the B-spline basis increases exponentially with the order.

B-splines were first introduced by Schoenberg in ([4], [17]) . A nice compendium of many of their algebraic properties can be found in [5]. These functions arealso known as hump functions, patch functions or hill functions. In this section, we list a few facts about B-splines for later reference.

B-splines were first introduced by Schoenberg in ([4], [,17]. Various algebraic properties of B spline can be found in [21]. These functions are also known as hump functions, patch functions or hill functions. In many formulations, we use B-splines for their smoothness as well as their finite support, which makes them computationally efficient and easy to implement.

Let $N_{i,k}$ represents B-spline of order k and it is partitioned on t, where k is positive real number and $t=(t_i)$ be a non decreasing sequence of real numbers. From [22], we can know that B spline $\{N_{i,k}\}$ are linearly independent if and only if $t_i < t_{i+k}$ for all i. The relation between B-splines can be given as,

$$N_{i,k}(t) = (t - t_i)\beta_{i,k-1}(t) + (t_{i+k} - t)\beta_{i+1,k-1}(t) \qquad (12)$$

$$\beta_{i,k}(t) = \begin{cases} N_{i,k}(t)/(t_{i+k} - t_i) & if \quad t_i \neq t_{i+k} \\ 0 & , \quad Otherwise \end{cases} \qquad (13)$$

and,

$$N_{i,1} = \begin{cases} 1 & if \quad t_i < t < t_{i+1} \\ 0 & , \quad Otherwise \end{cases} \qquad (14)$$

The sum of splines can also be normalised to 1.
Hence we can construct B-splines using equations (12, 13, 14) and feed non uniform samples directly to it so the resultant will be non uniformly interpolated signal. And if we reconstruct such a signal, the simulated results are as given in Fig.3. .Here we are directly interpolating the non-uniform samples with the help of spline and then reconstructing it. Another kind of approach that can be used is to use non-uniform splines to approximate the Lagrange's kernel $G(t)$.

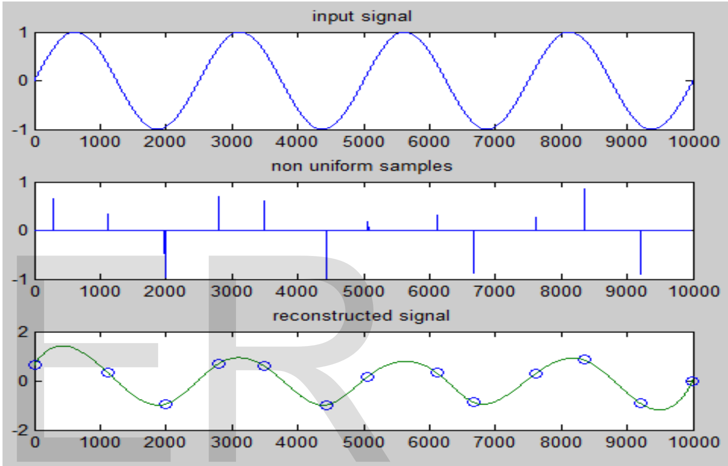

Fig.3 Reconstruction from NU samples using method of spline

4. ERROR COMPARISON OF APPROXIMATION METHODS :
Here we can compare the reconstruction results of Fig. 1 and Fig.3. From these simulation results we cannot analyse which method is more accurate. So for this we will compare performance of both methods by numerical experiments. And for that, we will calculate various errors like Mean Squared error and Mean absolute deviation.

Normalized mean squared error (MSE) of the output is given by,

$$MSE = \frac{\int_{1}^{-1} \left| \hat{x}(t) - x(t) \right|^2 dt}{\int_{1}^{-1} \left| x(t) \right|^2 dt}, \qquad (15)$$

Mean absolute deviation is a measure of dispersion, a measure of by how much the values in the data set are likely to differ from their mean. The absolute value is used to avoid deviations with opposite signs cancelling each other out.

The mean absolute deviation can be given by formula

$$MAD = \frac{1}{n}\sum_{i=1}^{n}\left|x_i - \overline{x}\right|, \qquad (16)$$

where n is the number of observed values, x-bar is the mean of the observed values and x_1 are the individual values.

Fig. 4 (ii) shows Mean Squared error (MSE) for reconstruction using approximation methods like sinc interpolation and method of spline. The error has been calculated for 100 instances and we can observe in the Fig.4 (ii) that MSE_Sinc is less than MS_spline .Numerically, MSE_sinc is observed in the range of 0.02 to 0.05. While MSE_spline is having somewhat higher values i.e in the range of 0.02 to 0.07 .While in Fig.4 (i), we can observe the behaviour of Mean Absolute Deviation (MAD) also. It is also calculated for 100 events and we can observe results similar MSE, MAD_Spline is having greater deviation than MAD_Sinc .MAD_Sinc is in the range of 0.03 to 0.09 and MAD_Spline 0.04 to 0.15.

Fig.5 shows comparison of average MSE which is calculated for 100 instances. We iterated 25 times to observe consistency of results. From which we conclude that the method of approximation using Sinc is better than that of spline. Fig.6 shows Mean absolute deviations for 25 iterations which are again calculated by averaging 100 instances. And here also we can say that deviation is observed more for the method of splines.

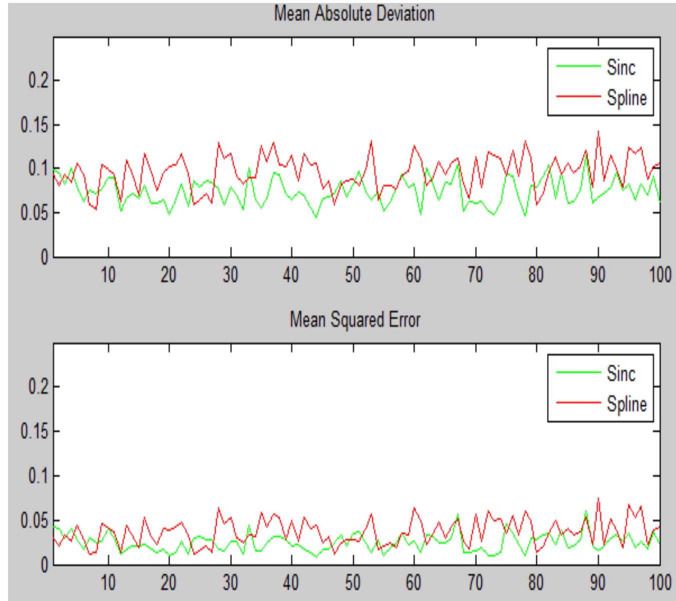

Fig. 4 (i) Mean absolute deviation comparison of reconstruction from NU samples using sinc interpolation deviation comparison
(ii) Mean squared error comparison of reconstruction from NU samples using sinc interpolation deviation comparison

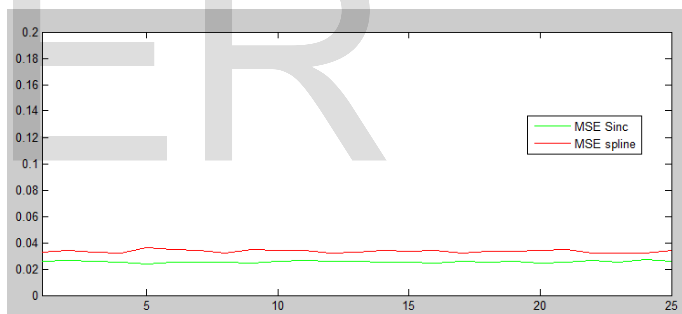

Fig. 5 Mean squared error comparison for 25 iterations

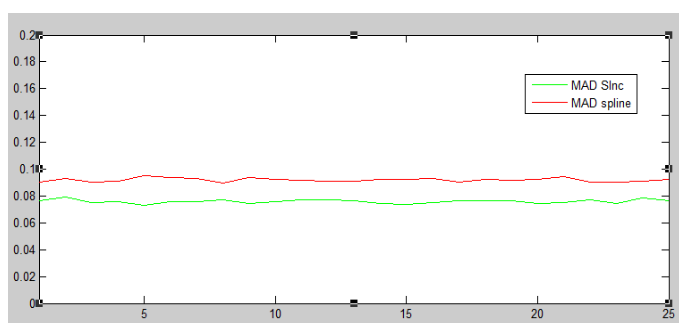

Fig. 6 Mean absolute deviation for 25 iterations

5. CONCLUSION :

From all above discussion we can conclude that if we observe both of the approximation methods, we can say that approximation using sinc interpolation method can reconstruct the signal from non-uniform samples more accurately than method of splines. Hence the performance of approximation methods can be measured in the form of errors in the reconstruction with respect to input.

For the future study, we include the reconstruction of non-uniform samples using the weighted splines. According to analysis provided uptill weighted splines may lead to better reconstruction than both of the methods.

REFERENCES :

[1] Aldroubi, A. and Grochenig, K., 2000, *J. Fourier Anal. Appl.*, **6** (1), 93–103.

[2] Black, W. C. and Hodges, D. A., 1980, *IEEE Journal of Solid-State Circuits*, **15** (6), 1022–1029.

[3] Boor, C. D., 1970, *J. Approximat. Theory*, **6**, 50-62.

[4] Curry, H. B. and Schoenberg, I. J., 1947, *Bull. Amer. Math. Sot.*, **53**, 1114.

[5] Curry H. B. and Schoenberg, I. J., 1966, *J. Anal. Math.*, **17**, 71-107.

[6] Eldar, Y. C. and Oppenheim, A. V., 2000, *IEEE Trans. Signal Processing*, **48**, (10), 2864–2875.

[7] Grochenig, K.and Schwab, H., 2003, *Matrix Anal. and Appl.*, **24** (4), 899–913.

[8] Grochenig, K. and Razafinjatovo, H., 1996, *J. Lond. Math. Soc.*, **54** (3), 557–567.

[9] Higgins, J. R., 1976, " *IEEE Trans.Information Theory*, **22** (5), 621-622.

[10] Jerri, A. J. , *Proceedings of the IEEE*, **65** (11), 1565–1596.

[11] Johansson, H. and Löwenborg, P., 2002, *IEEE Transactions on Signal Processing*, **50** (11), 2757–2767.

[12] Levinson, N., 1940, *Gap and Density Theorems, New York: A.M.S*.

[13] Maymon, S. and Oppenheim, A. V., 2010, *ICASSP-2010, IEEE International Conference on Acoustics, Speech and Signal Processing*.

[14] Mitra, S. K. and Kaiser, J. F., 1993, *Handbook for digital signal processing*, Wiley.

[15] Papoulis, A., 1977, *IEEE Trans. Circuits and Systems*, vol. CAS-24, No. 11, 652–654.

[16] Prendergast, R. S., Levy, B. C. and Hurst, P. J., 2004, *IEEE Transactions on Circuits and System I*, **51** (8), 1612–1622.

[17] Schoenbergc, I. J., 1946, *Quart. Appl. Math.* **4**, 45-99; 112-141.

[18] Strohmer, T. and Tanner, J., 2006, *M J. Numer. Anal.*, 44 (3), 1073– 1094.

[19] Wei, D., 2007, " *Proceedings of the 41st Annual Asilomar Conference on Signals, Systems, and Computers (Asilomar, CA)*, 4-7.

[20] Yen, J. L. 1956, *IRE Trans. Circuit Theory*, **3** (4) , 251–257.

[21] Yao, K. and Thomas, J. B., 1967, *IEEE Trans. Circuit Theory*, CT-14 (4) , 404–408.